Quantum algebras in phenomenological description of particle properties

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Quantum and q-deformed algebras find their application not only in mathematical physics and field theoretical context, but also in phenomenology of particle properties. We describe (i) the use of quantum algebras $U_q(su_n)$ corresponding to Lie algebras of the groups SU_n, taken for flavor symmetries of hadrons, in deriving new high-accuracy hadron mass sum rules, and (ii) the use of (multimode) q-oscillator algebras along with q-Bose gas picture in modelling the properties of the intercept λ of two-pion (two-kaon) correlations in heavy-ion collisions, as λ shows sizable observed deviation from the expected Bose-Einstein type behavior. The deformation parameter q is in case (i) argued and in case (ii) conjectured to be connected with the Cabibbo angle θ_{C}.

1. Introduction

Quantum groups and quantum or q-deformed algebras [1,2], whose basic mathematical aspects and diverse quantum physical applications are intensively studied for about a decade and half, will belong to most important and perspective tools of research in the 3rd millenium, too.

In this talk, meant as a mini-review, we concentrate on two examples of applying q-algebras to phenomenology of hadrons. Within the first one initiated in [3] and developed in subsequent papers, the q-analogs $U_q(su_n)$ of the Lie algebras of groups SU_n are adopted for hadronic flavor symmetries in order to derive new results concerning hadron masses and mass sum rules. Basic tool of this aproach is the representation theory of the q-algebras $U_q(su_n)$ [1,2], and in sections 2-8 we discuss a number of results, including unexpected implications: a possibility (gained due to use of q-algebras) to label different flavors topologically - by knot invariants; a direct link of deformation parameter to the Cabibbo angle [4], etc. In the second part of the talk (section 9) we consider a usage of the algebras of q-deformed oscillators, within a model of q-Bose gas, for effective description of unusual (non-Bose type) behavior of two-particle correlations of hadrons (pions or kaons) produced and registered in heavy ion collisions.

*This research was partly supported by the U.S. Civilian Research and Development Foundation Grant UP1-2115.

2. Vector mesons: q-deformation vs mixing

We use (see [3,5,6]) Gelfand-Tsetlin basis vectors for meson states from (n^2-1)-plet of 'n-flavor' $U_q(u_n)$ embedded into $\{(n+1)^2-1\}$-plet of 'dynamical' $U_q(u_{n+1})$; construct mass operator $\hat{M}_n$, invariant under the 'isospin+hypercharge' q-algebra $U_q(u_2)$, from the generators of dynamical algebra $U_q(u_{n+1})$ (e.g., $\hat{M}_3 = M_0\mathbf{1} + \gamma_3 A_{34}A_{43} + \delta_3 A_{43}A_{34}$); calculate expressions for masses $m_{V_i} \equiv \langle V_i|\hat{M}_3|V_i\rangle$ - these involve M_0, the parameters γ_3, δ_3, and the q-parameter. In particular, for $n=3$ one obtains

$$m_\rho = M_0, \quad m_{K^*} = M_0 - \gamma_3,$$
$$m_{\omega_8} = M_0 - 2[2]_q/[3]_q\gamma_3, \qquad (1)$$

where $[x]_q \equiv [x] = \frac{q^x - q^{-x}}{q-q^{-1}}$ is the q-number 'deforming' a number x and, to have equal masses for particles/antiparticles, $\delta_3 = \gamma_3$ was set. q-Dependence appears only in the mass of ω_8 (isosinglet in $U_q(su_3)$-octet). Excluding M_0, γ_3, the q-analog of Gell–Mann - Okubo (GMO) relation is [3] :

$$[3]_q m_{\omega_8} + (2[2]_q - [3]_q)m_\rho = 2[2]_q m_{K^*} \ . \qquad (2)$$

In the limit $q=1$ (then, $\frac{[3]_q}{[2]_q} = \frac{3}{2}$), this reduces to usual GMO formula $3m_{\omega_8} + m_\rho = 4m_{K^*}$ which needs singlet mixing [7]. However, it also yields

$$m_{\omega_8} + m_\rho = 2m_{K^*} \qquad \text{if} \quad q = e^{i\pi/5} \tag{3}$$

(then, $[3]_q = [2]_q$). With $m_{\omega_8} \equiv m_\phi$, and no mixing, eq.(3) coincides with nonet mass formula of Okubo [8] agreeing *ideally* with data [9]. The deformation angle $\frac{\pi}{5}$, see (3), coincides remarkably with ω-ϕ mixing angle (known [9] to be $\theta_{\omega\phi} = 36°$) of traditional $SU(3)$-based scheme. In other words, the q-deformation of flavor symmetries *supersedes* the issue of singlet mixing.

For $3 < n \le 6$ the scheme works as well. Again, only masses of singlets ω_{15}, ω_{24}, ω_{35} from (n^2-1)-plets of $U_q(u_n)$ involve q-dependence. As result, we get the q-analog (with isodoublet D^*)

$$\frac{[4]}{[3]} m_{\omega_{15}} + \left(2\frac{[4]^2}{[3]^2} - \frac{8}{[2]}\frac{[4]}{[3]} - \frac{[4]}{[3]} + 4\right)\ m_\rho = \\ = 2\ m_{D^*} + \left(2\frac{[4]^2}{[3]^2} - \frac{8}{[2]}\frac{[4]}{[3]} + 2\right) m_{K^*}, \tag{4}$$

and q-analogs for n=5, 6 (see [3,5,6]). Fixing q by setting $[4]_q$=$[3]_q$ (and $[n]_q$=$[n-1]_q$, n=5, 6) oversimplifies the relations and yields higher analogs of Okubo's nonet sum rule (isodoublets in r.h.s):

$$m_{\omega_{15}} + (5 - 8/[2]_{q_4}) m_\rho = \\ = 2\ m_{D^*} + (4 - 8/[2]_{q_4}) m_{K^*}, \tag{5}$$

$$m_{\omega_{24}} + (9 - 16/[2]_{q_5}) m_\rho = \\ = 2\ m_{D_b{}^*} + (4 - 8/[2]_{q_5})(m_{D^*} + m_{K^*}), \tag{6}$$

$$m_{\omega_{35}} + (13 - 24/[2]_{q_6}) m_\rho = \\ = 2\ m_{D_t{}^*} + (4 - 8/[2]_{q_6})(m_{D_b{}^*} + m_{D^*} + m_{K^*}). \tag{7}$$

Here the values $q_n = e^{i\pi/(2n-1)}$ (for which $[2]_{q_n} = 2\cos\frac{\pi}{2n-1}$) solve eqs. $[n]_q - [n-1]_q = 0$. Like in the case with $m_{\omega_8} \equiv m_\phi$, it is meant in (5)-(7) that J/ψ is put in place of ω_{15}, Υ in place of ω_{24}, toponium in place of ω_{35} (i.e., no mixing!). With experimental masses, eq.(5) holds to within 2.6% and eq.(6) holds with precision $\simeq 0.7\%$.

The q-polynomials $[n]_q - [n-1]_q$ have a topological meaning.

Labelling flavors by knots invariants

Polynomials $[n]_q - [n-1]_q \equiv P_n(q)$, by their roots, reduce q-analogs (2), (4) and those for n=5, 6 to realistic mass sum rules (MSR) (3), (5)-(7). And, due to the property (i) $P_n(q) = P_n(q^{-1})$, (ii) $P_n(1) = 1$, they coincide [3,6] with such knot invariants as Alexander polynomials $\Delta_q\{(2n-1)_1\}$ of $(2n-1)_1$-torus knots. E.g., $[3]_q - [2]_q = q^2 + q^{-2} - q - q^{-1} + 1 \equiv \Delta_q\{5_1\}$, $[4]_q - [3]_q = q^3 + q^{-3} - q^2 - q^{-2} + q + q^{-1} - 1 \equiv \Delta_q\{7_1\}$ correspond to the 5_1- and 7_1-knots. Since the extra q-deuce in (4) can be linked to the trefoil (or 3_1-) knot: $[2]_q - 1 = q + q^{-1} - 1 \equiv \Delta_q\{3_1\}$, *all the q-dependence* in masses of ω_{n^2-1}, in coefficients of (2),(4) and of higher q-analogs, is expressible through Alexander polynomials. Namely, $\frac{[3]_q}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{\Delta\{3_1\}+1}$, $\frac{[4]_q}{[3]_q} = 1 + \frac{\Delta\{7_1\}}{[3]_q} = 1 + \frac{\Delta\{7_1\}}{\Delta\{5_1\}+\Delta\{3_1\}+1}$, etc. The values q_n are thus roots of respective Alexander polynomials. For each n, just the 'senior' (numerator) polynomial in $\frac{[3]_q}{[2]_q}$, $\frac{[4]_q}{[3]_q}$ and $\frac{[n]_q}{[n-1]_q}$, $n = 5, 6$, serves to 'single out', by its root, the corresponding MSR from q-deformed analog.

Thus, the q-parameter for each n is fixed *rigidly* as a root q_n of $\Delta\{(2n-1)_1\}$, contrary to choice of q by fitting in other phenomenological applications [10]. Here, using flavor q-algebras along with 'dynamical' q-algebras according to $U_q(u_n) \subset U_q(u_{n+1})$, we gain: the torus knots 5_1, 7_1, 9_1, 11_1 are put into correspondence [5,6] with vector quarkonia $s\bar{s}$, $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ respectively. The polynomial $P_n(q) \equiv [n]_q - [n-1]_q$ by its root $q_n = q(n)$ determines the value of q-parameter for each n and thus serves as *defining polynomial* for the MSR/quarkonium/flavor corresponding to n. Hence, the use of q-algebras suggests a possibility of *topological labeling of flavors*: fixed number n corresponds to $2n-1$ overcrossings of 2-strand braids whose closure gives these $(2n-1)_1$-torus knots. With the form $(2n-1, 2)$ of same torus knots this means the correspondence $n \leftrightarrow w \equiv 2n-1$, w being the winding number around one of the two basic cycles on torus.

3. Octet baryon mass sum rules: best candidates from q-deformation

Using $U_q(su_n)$, the q-deformed mass relation

$$[2]M_N + \frac{[2]M_\Xi}{[2]-1} = [3]M_\Lambda + \left(\frac{[2]^2}{[2]-1} - [3]\right)M_\Sigma \\ + \frac{A_q}{B_q}\left(M_\Xi + [2]M_N - [2]M_\Sigma - M_\Lambda\right) \tag{8}$$

was obtained [5,6] where A_q, B_q are certain polynomials of $[2]_q$ with non-overlapping sets of zeros. This q-analog yields, as three particular cases, the familiar Gell-Mann - Okubo mass relation (in the 'classical' case of $q=1$) and two new MSRs of improved accuracy [5,6,11]:

$$M_N + M_\Xi = \tfrac{3}{2}M_\Lambda + \tfrac{1}{2}M_\Sigma, \qquad (0.58\%) \quad (9)$$

$$M_N + \tfrac{1+\sqrt{3}}{2}M_\Xi = \tfrac{2M_\Lambda}{\sqrt{3}} + \tfrac{9-\sqrt{3}}{6}M_\Sigma, \ (0.22\%) \ (10)$$

$$M_N + \tfrac{M_\Xi}{[2]_{q_7}-1} = \tfrac{M_\Lambda}{[2]_{q_7}-1} + M_\Sigma. \qquad (0.07\%) \quad (11)$$

Different dynamical representations, after calculation, produce in (8) differing pairs A_q, B_q. Each A_q contains the factor $([2]_q - 2)$ i.e., the 'classical' zero $q=1$, and some other nontrivial zeros. Eqs. (10), (11) result from two different dynamical representations $D^{(1)}$ resp. $D^{(2)}$ producing $A_q^{(1)}$ resp. $A_q^{(2)}$ which possess zeros $q_6 = e^{i\pi/6}$ resp. $q_7 = e^{i\pi/7}$. The choise (11), i.e. $q_7 = e^{i\pi/7}$, provides the best mass sum rule.[2]

Sum rule (10) was first derived [5] from a dynamical representation (irrep) $D^{(1)}$ of $U_q(u_{4,1})$. However, the 'compact' dynamical $U_q(u_5)$ is equally well suited. Among the admissible dynamical irreps there exist an entire series of irreps (numbered by integer m, $6 \le m < \infty$) which produce infinite set of MSRs, each given by the first line in (8) with q_m put for q, where $q_m = e^{i\pi/m}$ guarantees vanishing of $\frac{A_q}{B_q}$. *Each of these MSRs shows better agreement with data* than the classical GMO one. To illustrate, few cases from the infinite set are shown in the table, the 1st row of which being the classical GMO with $q_\infty = 1$.

$\theta = \frac{\pi}{m}$	(RHS−LHS), MeV	$\frac{\lvert\text{RHS}-\text{LHS}\rvert}{\text{RHS}}$, %
π/∞	26.2	0.58
$\pi/30$	25.42	0.56
$\pi/12$	20.2	0.44
$\pi/8$	10.39	0.23
$\pi/7$	3.26	0.07
$\pi/6$	-10.47	0.22

We thus gain that a 'discrete choice' becomes possible instead of usual fitting; the q-polynomials A_q due to zeros q_m serve as *defining* polynomials for the corresponding MSRs.

Quark mass ratio in terms of baryon masses

Since $[2]_{q_7} = 2\cos\frac{\pi}{7}$, the MSR (11) takes the equivalent form of "modified average"

$$\frac{M_\Xi - M_N + M_\Sigma - M_\Lambda}{2\cos(\pi/7)} = M_\Sigma - M_N. \qquad (12)$$

From (12) with $\frac{\pi}{7} = 2\theta_C$ (see footnote 2), using the famous relation [13] $\tan^2\theta_C = m_d/m_s$ we infer a new formula giving quark mass ratio in terms of (very precisely known) octet baryon masses:

$$\frac{m_s}{m_d} = \frac{3M_\Sigma - M_\Lambda - 3M_N + M_\Xi}{M_\Sigma + M_\Lambda - M_N - M_\Xi} = 18.63 \pm 0.16.$$

Numerically, the obtained ratio is in nice agreement with the value $\frac{m_s}{m_d}$=18.9 ± 0.8 given in [14].

4. Mass sum rules for decuplet baryons: the q-analog matches empirical masses

In the case of $SU(3)$-decuplet baryons $\frac{3}{2}^+$, 1st order symmetry breaking yields [7] equal spacing rule (ESR) for isoplet members in **10**-plet. Empirically, $M_{\Sigma^*} - M_\Delta$, $M_{\Xi^*} - M_{\Sigma^*}$ and $M_\Omega - M_{\Xi^*}$ show sensible deviation from ESR: 152.6 $MeV \leftrightarrow$ 148.8 $MeV \leftrightarrow$ 139.0 MeV. The other mass relation known long ago [15],

$$(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/2 = M_{\Xi^*} - M_{\Sigma^*}, (13)$$

accounts 1st and 2nd order of $SU(3)$-breaking and holds only slightly better than the ESR.

On the contrary, use of the q-algebras $U_q(su_n)$ instead of $SU(n)$ leads to sizable improvement. From evaluations of decuplet masses in two particular irreps of the dynamical algebra $U_q(u_{4,1})$, the q-deformed mass relation

$$\frac{M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*}}{2\cos\theta_{\mathbf{10}}} = M_{\Xi^*} - M_{\Sigma^*} \qquad (14)$$

was derived [16]. As proven there, this mass relation is <u>universal</u> - it results from any admissible irrep (which contains $U_q(su_3)$-decuplet embedded in *20*-plet of $U_q(su_4)$) of the dynamical $U_q(u_{4,1})$. With empirical masses [9], the formula (14) holds remarkably for $\theta_{\mathbf{10}} \simeq \frac{\pi}{14}$ (in fact, $\theta_{\mathbf{10}} = \theta_C$, see footnote 2 and sec. 7 below).

[2] The value q_7 is linked [6,12] to the Cabibbo angle: $\frac{1}{i}\ln q_7 \equiv \theta_{\mathbf{8}} = \frac{\pi}{7} = 2\theta_C$ (see also sec. 7 below).

The universality of q-analog (14) extends also to all admissible irreps of the *'compact'* dynamical $U_q(su_5)$. Say, within a dynamical irrep $\{4000\}$ of $U_q(su_5)$ calculation yields: $M_\Delta = M_{\mathbf{10}} + \beta$, $M_{\Sigma^*} = M_{\mathbf{10}}+[2]\beta+\alpha$, $M_{\Xi^*} = M_{\mathbf{10}}+[3]\beta+[2]\alpha$, $M_\Omega = M_{\mathbf{10}} + [4]\beta + [3]\alpha$, from which (14) stems. With hypercharge Y, all four masses are comprised by single formula for $M_{D_i} \equiv M\big(Y(D_i)\big)$:

$$M_{D_i}{=}M_{\mathbf{10}}{+}\alpha[1{-}Y(D_i)]_q + \beta[2{-}Y(D_i)]_q, \quad (15)$$

with explicit dependence on Y. If $q = 1$, this reduces to $M_{D_i} = \tilde{M}_{\mathbf{10}} + a\ Y(D_i)$, i.e. *linear dependence on hypercharge* (or strangeness) where $a = -\alpha - \beta$, $\tilde{M}_{\mathbf{10}} = M_{\mathbf{10}} + \alpha + 2\beta$.

5. Highly nonlinear $SU(3)$-breaking effects in baryon masses

Formula (15) involves *highly nonlinear dependence* of mass on hypercharge (for decuplet, Y alone causes $SU(3)$-breaking). Since for the q-number $[N]_q$ we have $[N]_q = q^{N-1}+q^{N-3}+\ldots+q^{-N+3}+q^{-N+1}$ (N terms), this shows exponential Y-dependence of masses. Such high nonlinearity makes (14) and (15) radically different from the result (13) of traditional treatment accounting linear and quadratic effects in Y.

For octet baryon masses, *nonpolynomiality* in $SU(3)$-breaking effectively accounted by the model was explicitly shown in [11]. For this, one analyses the expressions for isoplet masses with explicit dependence on hypercharge Y and isospin I, through $I(I+1)$. Typical matrix element contributing to octet baryon masses contains terms such as $([Y/2]_q[Y/2{+}1]_q - [I]_q[I{+}1]_q)$ or $([Y/2-1]_q[Y/2-2]_q - [I]_q[I+1]_q)$ (with multipliers depending on irrep labels m_{15}, m_{55}), which show explicitly the dependence on hypercharge and the factor $[I]_q[I+1]_q$ q-deforming the $SU(2)$ Casimir. From definition of q-bracket $[n] = \frac{\sin(nh)}{\sin(h)}$, $q{=}\exp(ih)$, it is clearly seen that baryon masses depend on hypercharge Y and isospin I (hence, on $SU(3)$-breaking effects) in highly nonlinear - *nonpolynomial* - fashion.

The ability to account highly nonlinear $SU(3)$-breaking effects by applying quantum analogs $U_q(su_n)$ of usual flavor symmetries is much alike the fact shown in [17] that, by exploiting appropriate *free* q-deformed structure one is able to efficiently study the properties of (undeformed) quantum-mechanical systems with complicated interactions.

6. Using the Hopf-algebra structure

As demonstrated, our approach supplies a plenty of q-analogs (with different pairs A_q, B_q) of the form (8). A completely different, as regards (8), version of q-deformed analog can be derived [11] using for the symmetry breaking term in mass operator a component of *q-tensor operator*. This implies usage of the Hopf algebra structure (comultiplication, antipode) of the quantum algebras $U_q(su_n)$, through the q-tensor operators (V_1, V_2, V_3) resp. $(V_{\bar 1}, V_{\bar 2}, V_{\bar 3})$ formed from elements of $U_q(su_4)$ and transforming as $\mathbf{3}$ resp. $\mathbf{3}^*$ under the adjoint action of $U_q(su_3)$. With the Cartan elements H_1, H_2, denoting $[X,Y]_q{\equiv}XY{-}qYX$, the components (V_1, V_2, V_3) and $(V_{\bar 1}, V_{\bar 2}, V_{\bar 3})$ can be found explicitly [11], e.g., $V_2 = [E_2^+, E_3^+]_q q^{H_1/6-H_2/6}$, $V_3 = E_3^+ q^{H_1/6+H_2/3}$, and $V_{\bar 3} = q^{H_1/6+H_2/3}E_3^-$. The mass operator is given as

$$\begin{aligned}\hat{M} &= \hat{M}_0 + \hat{M}_8 = M_0\mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)} \\ &= M_0\mathbf{1} + \alpha V_3 V_{\bar 3} + \beta V_{\bar 3} V_3\end{aligned}$$

where $\hat{M}_0$ is $U_q(su_3)$-invariant and $\hat{M}_8$ transforms as $I{=}0, Y{=}0$ component of tensor operator of $\mathbf{8}$-irrep of $U_q(su_3)$, and it is taken into account that the irrep $\mathbf{8}$ occurs twice in the decomposition of $\mathbf{8}\otimes\mathbf{8}$. Besides, the isosinglet operators $V_3V_{\bar 3}$ and $V_{\bar 3}V_3$ arise in accordance with $\mathbf{3}\otimes\mathbf{3}^* = \mathbf{1}\oplus\mathbf{8}$, $\mathbf{3}^*\otimes\mathbf{3}{=}\mathbf{1}\oplus\mathbf{8}$.

The final form of mass operator, with redefined M_0, α, β, is

$$\hat{M} = M_0\mathbf{1} + \alpha E_3^+ E_3^- q^Y + \beta E_3^- E_3^+ q^Y \quad (16)$$

where the hypercharge $Y = (H_1 + 2H_2)/3$. Matrix elements with $\hat{M}$ from (16) are evaluated by embedding $\mathbf{8}$ in a particular irrep of $U_q(su_4)$. Evaluation of baryon masses, say, within the irrep $\mathbf{15}$ of $U_q(su_4)$ yields: $M_N = M_0 + \beta q$, $M_\Sigma = M_0$, $M_\Lambda = M_0 + \frac{[2]}{[3]}(\alpha+\beta)$, $M_\Xi = M_0 + \alpha q^{-1}$. Excluding M_0, α, β, one finds:

$$[3]M_\Lambda + M_\Sigma = [2](q^{-1}M_N + qM_\Xi). \quad (17)$$

The q-parameter now *can be fixed by a fitting only* and, for each of values $q_{1,2} = \pm 1.035$, $q_{3,4} = \pm 0.903\sqrt{-1}$, the q-MR (17) indeed holds within experimental uncertainty.

7. The link: q-parameter $\leftrightarrow$ Cabibbo angle

For pseudoscalar (PS) mesons, the generalization [18] of GMO-formula

$$f_\pi^2 m_\pi^2 + 3 f_\eta^2 m_\eta^2 = 4 f_K^2 m_K^2 \tag{18}$$

involves decay constants as coefficients. On imposing the constraint[3] $f_\pi^2 + 3f_\eta^2 = 4f_K^2$ it becomes

$$m_\pi^2 + \frac{9 f_K^2/f_\pi^2}{4 - f_K^2/f_\pi^2} m_\eta^2 = 4 \frac{f_K^2}{f_\pi^2} m_K^2, \tag{19}$$

to be compared with our q-*analog* (2) rewritten for PS mesons (with masses squared):

$$m_\pi^2 + \frac{[3]}{2[2]-[3]} m_{\eta_8}^2 = \frac{2[2]}{2[2]-[3]} m_K^2. \tag{20}$$

This holds for (the mass of) *physical* η-meson put for η_8 (i.e., no mixing), *at properly fixed* $q = q_{\rm PS}$.

The two extensions (19) resp. (20) both reduce to standard GMO in the corresponding limits $\frac{f_K}{f_\pi} \to 1$ resp. $q \to 1$. From the identification

$$f_K^2/f_\pi^2 \;\leftrightarrow\; \frac{1}{2}[2]/(2[2]-[3]), \tag{21}$$

using $[3]_q = [2]_q^2 - 1$ and the notation $\xi_{\pi,K} \equiv (4f_K^2/f_\pi^2)^{-1}$, we get

$$[2]_\pm = 1 - \xi_{\pi,K} \pm \sqrt{\left(1 - \xi_{\pi,K}\right)^2 + 1}. \tag{22}$$

The ratio f_K/f_π is expressible through the Cabibbo angle, e.g., by the formula $\tan^2\theta_{\rm C} = \frac{m_\pi^2}{m_K^2}\left[\frac{f_K}{f_\pi} - \frac{m_\pi^2}{m_K^2}\right]^{-1}$ (see [19]). With (21), (22) this implies: the *deformation parameter* $q_{\rm PS}$ *is directly connected with the Cabibbo angle.*

One can arrive at such conclusion in another way. In [20], the q-deformed lagrangian for gauge fields of the Weinberg - Salam (WS) model, invariant against quantum-group valued gauge transformations, was constructed. The formula $F^0_{\mu\nu} = {\rm Tr}_q(F_{\mu\nu})\,[2(q^2+q^{-2})]^{-1/2} = B_{\mu\nu}\cos\theta + F^3_{\mu\nu}\sin\theta$ obtained therein, along with expression for $F^3_{\mu\nu}$ and the relation

$$\tan\theta = (1-q^2)/(1+q^2), \tag{23}$$

exhibits mixing of the $U(1)$-component B_μ with third (nonabelian) component A^3_μ. Forming new potentials $\tilde{A}_\mu = B_\mu\cos\theta + A^3_\mu\sin\theta$, $Z_\mu = -B_\mu\sin\theta + A^3_\mu\cos\theta$ yields physical photon $\tilde{A}_\mu$ and Z-boson of WS model, where $\theta = \theta_{\rm W}$, i.e., the Weinberg angle (at $\theta = 0$ the potentials B_μ and A^3_μ get completely unmixed whereas nonzero θ, i.e., nontrivial q-deformation provides proper mixing inherent for the WS model). To summarize: *weak mixing is adequately modelled by the q-deformation.* That is, the q-deformation is able to realize proper weak mixing of gauge fields and provides explicit connection of the weak angle and the deformation parameter q, see eq.(23).

On the other hand, the relation found in [21]

$$\theta_{\rm W} = 2(\theta_{12} + \theta_{23} + \theta_{13}) \tag{24}$$

connects $\theta_{\rm W}$ with the Cabibbo angle $\theta_{12} \equiv \theta_{\rm C}$ (and the angles θ_{13}, θ_{23} of mixing with 3rd family, that will be neglected). The eqn. (24) is important as it links apparently different mixings: in the *bosonic* (interaction) versus *fermionic* (matter) sectors of the electroweak model.

Combining (23) and (24) we conclude: the Cabibbo angle can be linked with q-parameter of a quantum-group (or quantum-algebra) based structure *applied in the fermion sector.* Hence, there must exist a direct connection of the q-parameter in (12), (14) with the fermion mixing angle. Setting $\theta_{\mathbf{10}} = g(\theta_{\rm C})$ and $\theta_{\mathbf{8}} = h(\theta_{\rm C})$ we find for $g(\theta_{\rm C})$ and $h(\theta_{\rm C})$ the following:

$$\theta_{\mathbf{10}} = \theta_{\rm C}, \qquad \theta_{\mathbf{8}} = 2\,\theta_{\rm C}. \tag{25}$$

With $\theta_{\mathbf{8}} = \frac{\pi}{7}$, see (12), this suggests for Cabibbo angle the exact value $\frac{\pi}{14}$.

Cabibbo mixing from noncommutative extra dimensions?

Quantum groups and the related quantum algebras provide necessary tools in constructing covariant differential calculi and particular noncommutative geometry on quantum spaces [2], e.g,

[3] It leads to the single dimensionless quantity $\frac{f_K}{f_\pi}$ involved in the multipliers of masses.

quantum vector spaces, quantum homogeneous spaces.

The direct link found between the Cabibbo angle $\theta_{\rm C}{=}\frac{\pi}{14}$ and the q-parameter which measures strength of q-deformation for the q-algebras $U_q(su_n)$ used for flavor symmetry, can be viewed [12] as indicating towards noncommutative-geometric origin of fermion mixing. The exact value $\theta_{\rm C}{=}\frac{\pi}{14}$ of the Cabibbo angle would then serve as the *noncommutativity measure* of relevant quantum space, responsible for the mixing and explicitly as yet unknown, in extra dimensions whose number should be not less than 2.

8. Mass relations from anyonic realization of $U_q(su_N)$

Necessary setting adopted from [22] includes lattice angle functions $\theta_\gamma(\mathbf{x},\mathbf{y})$ and $\theta_\delta(\mathbf{x},\mathbf{y})$ for the two opposite (γ- and δ-) types of cuts and the related definition of ordering of sites on the lattice ($\mathbf{x} > \mathbf{y}$ or $\mathbf{y} > \mathbf{x}$). Accordingly, the two types of statistical operator, $K_i(\mathbf{x}_\gamma)$ and $K_i(\mathbf{x}_\delta)$, are formed using N sorts of lattice fermions $c_i(\mathbf{x})$, $c_i^\dagger(\mathbf{x}),\ \ i = 1, ..., N$, obeying usual (lattice) anti-commutation relations (ARs), as

$$K_j(\mathbf{x}_\gamma) = \exp\bigl(\mathrm{i}\nu \sum_{\mathbf{y}\neq\mathbf{x}} \theta_\gamma(\mathbf{x},\mathbf{y}) c_j^\dagger(\mathbf{y}) c_j(\mathbf{y})\bigr) \qquad (26)$$

and similarly for $K_i(\mathbf{x}_\delta)$. In terms of them, the two types of anyonic oscillators are given as [22]

$$a_i(\mathbf{x}_\gamma) = K_i(\mathbf{x}_\gamma) c_i(\mathbf{x}), \qquad a_i(\mathbf{x}_\delta) = K_i(\mathbf{x}_\delta) c_i(\mathbf{x}).$$

The relations of permutation (PRs) obtained for anyonic oscillators include simple ARs, and also nontrivial PRs involving the deformation parameter q (the latter is connected with the statistics parameter ν in eq. (26) as: $q = \exp(\mathrm{i}\pi\nu)$). For instance, the braiding properties are described by the following nontrivial PRs ($\mathbf{x} \neq \mathbf{y}$):

$$a_i(\mathbf{x}_\gamma) a_i(\mathbf{y}_\gamma) + q^{-\mathrm{sgn}(\mathbf{x}-\mathbf{y})} a_i(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0,$$
$$a_i(\mathbf{x}_\gamma) a_i^\dagger(\mathbf{y}_\gamma) + q^{\mathrm{sgn}(\mathbf{x}-\mathbf{y})} a_i^\dagger(\mathbf{y}_\gamma) a_i(\mathbf{x}_\gamma) = 0.$$

The basic fact proven in [22] states that generating elements $A_{j,j+1}$, $A_{j+1,j}$ and H_j realized bilinearly in terms of anyonic oscillators $a_i(\mathbf{x}_\gamma)$, $a_i^\dagger(\mathbf{y}_\gamma)$ satisfy the defining relations [1,2] of the quantum algebra $U_q(su_N)$. Similarly, dual realization in terms of $a_i(\mathbf{x}_\delta)$, $a_i^\dagger(\mathbf{y}_\delta)$ does also exist. On this basis, within anyonic realization of $U_q(su_N)$, one can explicitly construct [23] both basis vectors for hadronic irreps and hadron mass operator. Starting point is the heighest weight vector (HWV) of the irrep $\{4000\}$ of 'dynamical' $U_q(su_5)$ which is of the form $|1111\rangle$ in the notation $|n_1 n_2 n_3 n_4\rangle$ for the state vector, that means $a^\dagger_{n_1}(\mathbf{x}_{1\gamma}) a^\dagger_{n_2}(\mathbf{x}_{2\gamma}) a^\dagger_{n_3}(\mathbf{x}_{3\gamma}) a^\dagger_{n_4}(\mathbf{x}_{4\gamma})|0\rangle$. All basis state vectors of baryons $\frac{3}{2}^+$ are constructed, by acting with lowering generators, in accordance with the chain of q-algebras $U_q(su_3) \subset U_q(su_4) \subset U_q(su_5)$ and respective chain of irreps $[30] \subset [300] \subset \{4000\}$. For isoquartet baryon $|\Delta^{++}\rangle$ one finds $\frac{1}{\sqrt{[4]}}(|5111\rangle + q^{-1}|1511\rangle + q^{-2}|1151\rangle + q^{-3}|1115\rangle)$, and similarly for $|\Sigma^*\rangle$, $|\Xi^*\rangle$, $|\Omega^-\rangle$. The dual basis $|\widetilde{\Delta^{++}}\rangle$, etc., obtained by acting on the HWV with lowering operators in dual anyonic realization, is also needed. Masses M_{D_i} of baryons D_i are calculated within the dynamical $U_q(su_5)$-irrep $\{4000\}$ as $M_{D_i} = \widetilde{\langle D_i}|\hat{M}|D_i\rangle$ (with mass operator formed from anyonic operators) to yield: $M_\Delta = M_{\mathbf{10}}+\beta$, $M_{\Sigma^*} = M_{\mathbf{10}}+[2]_q\alpha+[2]_q\beta$, $M_{\Xi^*} = M_{\mathbf{10}} + [2]_q^2\alpha + [3]_q\beta$, and $M_{\Omega^*} = M_{\mathbf{10}} + [2]_q[3]_q\alpha + [4]_q\beta$. One easily checks that these masses satisfy the relation (14). This proves applicability [23] of quantum algebras and their irreps for treating hadron mass relations *within anyonic realization.*

9. Algebras of q-oscillators, q-Bose gas and two-pion (two-kaon) correlations

The model of ideal gas of q-bosons based on the algebra of q-deformed oscillators either of Biedenharn-Macfarlane (BM) type [24] or Arik-Coon (AC) type [25], was recently utilized within the approach aimed to describe [26,27] unusual properties of 2-particle correlations of identical pions or kaons produced in heavy ion collisions. The approach yields clear predictions based on explicit expressions for the intercept λ (dependent on temperature, particle mass, pair mean momentum, *and the deformation parameter* q).

To obtain needed observables, one evaluates thermal averages $\langle A\rangle = \mathrm{Sp}(A\rho)/\mathrm{Sp}(\rho)$, $\rho =$

$e^{-\beta H}$, where the Hamiltonian $H = \sum \omega_i N_i$ and $\beta = 1/T$. With $b_i^\dagger b_i = [N_i]_q$ and $q+q^{-1} = 2\cos\theta$, the q-deformed distribution function results for BM-type q-bosons as

$$\langle b_i^\dagger b_i \rangle = \frac{e^{\beta\omega_i} - 1}{e^{2\beta\omega_i} - 2\cos\theta e^{\beta\omega_i} + 1}. \tag{27}$$

At θ=0 (or q=1), it yields Bose-Einstein (B-E) distribution, since q=1 recovers usual bosonic commutation relations. As seen, deviation of q-distribution (27) from the quantum B-E distribution tends towards the Maxwell-Boltzmann one. This means reducing of quantum statistical effects. For kaons, whose mass $m_K > 3m_\pi$, analogous curve gets closer (than pion's one) to the B-E distribution. For AC-type q-bosons, the q-distribution is especially simple: $\langle b_i^\dagger b_i \rangle = \frac{1}{e^{\beta\omega_i} - q}$.

To obtain explicitly the intercept λ of two-particle correlations one starts with the defining ratio $\lambda + 1 = \langle b^\dagger b^\dagger b b \rangle / (\langle b^\dagger b \rangle)^2$, calculates the two-particle distribution $\langle b^\dagger b^\dagger b b \rangle$ and takes into account the $\langle b^\dagger b \rangle$. The result for AC-type q-bosons reads $\lambda = q - \frac{q(1-q^2)}{e^{\omega/T} - q^2}$, $-1 \le q \le 1$, and for BM-type q-bosons, with $\mathcal{F}(\beta\omega) \equiv \cosh(\beta\omega)$, it is

$$\lambda = -1 + \frac{2\cos\theta \ (\mathcal{F}(\beta\omega) - \cos\theta)^2}{(\mathcal{F}(\beta\omega) - 1)(\mathcal{F}(\beta\omega) - 2\cos^2\theta + 1)}. \tag{28}$$

Both (27), (28) are real owing to the sum $q + q^{-1}$.

The intercept λ, with $\omega = (m^2 + \mathbf{K^2})^{1/2}$, shows a remarkable feature: asymptotically, at large mean momentum of pion (kaon) pair and fixed temperature, λ tends to a constant given by the q-parameter: $\lambda^{\rm as} = q$ (q real) for the AC-type q-bosons, and

$$\lambda^{\rm as} = 2\cos\theta - 1, \qquad \theta = \frac{1}{i}\ \ln q, \tag{29}$$

for the BM-type q-bosons.

As conjectured in [27], correlations of pions and kaons are determined by the same value of q (a kind of universality). Then, experimentally one should observe the tendency of merging $\lambda(\pi)$ and $\lambda(K)$ at large enough mean momenta, i.e., $\lambda^{\rm as}(\pi) = \lambda^{\rm as}(K)$. Preliminary results of recent RHIC/STAR experiment give three values [28] $\lambda_1(\pi^-)$, $\lambda_2(\pi^-)$ and $\lambda_3(\pi^-)$ for the π^--intercept, obtained by averaging over three intervals of transverse momenta (in MeV/c): $(125 \div 225)$, $(225 \div 325)$, $(325 \div 450)$, and by integrating over rapidity in the range $-0.5 \le y \le 0.5$.

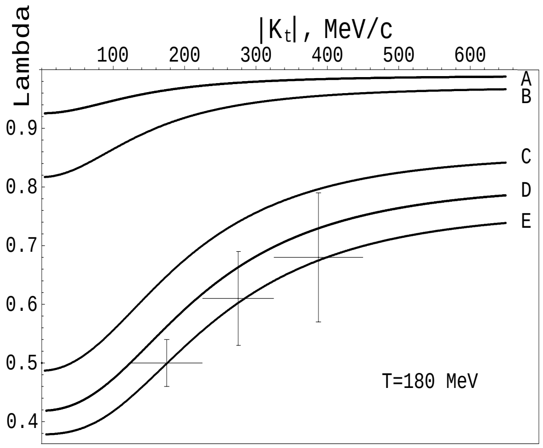

Figure 1. The transverse momentum $|\mathbf{K}_t|$ dependence of the intercept λ of two-pion correlation at fixed $T = 180$ MeV and fixed deformation parameter $q = \exp(i\theta)$: A) $\theta = 6°$, B) $\theta = 10°$, C) $\theta = 22°$, D) $\theta = 25.7°$ (i.e., $2\theta_{\rm C}$), E) $\theta = 28.5°$.

In Fig.1 the three values $\lambda_j(\pi^-)$, j=1, 2, 3, with error bars, are shown along with five curves for the intercept λ which correspond to fixing in (28) different values of the deformation angle θ, all curves being at the temperature $T = 180$ MeV. One can see remarkable agreement with data of the curve E obtained at $\theta = 28.5°$. The other interesting curve D corresponds to $\theta = \pi/7 \simeq 25.7°$ (twice the Cabibbo angle, see footnote 2 and eq.(25)). At the same temperature $T = 180$ MeV, the curve D agrees (within error bar) with the points $\lambda_2(\pi)$ and $\lambda_3(\pi)$. However, suffice it to take slightly higher effective temperature $T \simeq 198$ MeV, and the resulting curve marked by $\theta = \pi/7 = 2\theta_{\rm C}$ *respects all the three error bars*. Among different mixing angles known for hadrons, see [9], only the angle $2\theta_{\rm C}$ seems to be relevant to the discussed topic of intercept $\lambda(\pi)$. It is tempting to suggest that just this angle $2\theta_{\rm C}$ can be the benchmark of assumed universality (to be) seen in 2-particle correlations since, then, $\lambda^{\rm as}(\pi)|_{\theta=\pi/7} = \lambda^{\rm as}(K)|_{\theta=\pi/7} = 2\cos\frac{\pi}{7} - 1 = 0.80194$. Insisting on the asymptotical coincidence $\lambda^{\rm as}(\pi) = \lambda^{\rm as}(K)$ we may predict for kaon intercepts: *at any transverse momentum, the intercept $\lambda(K)$ of 2-kaon correlations should not exceed* 0.80194.

10. Outlook

A question naturally arises: does there exist more intimate connection between the two discussed applications - of the quantum algebras $U_q(su_n)$ taken as flavor symmetries, on one hand, and of the algebras of q-deformed oscillators corresponding to discretized momenta of (correlated) pairs of pions or kaons as produced in relativistic heavy ion collisions, on the other hand? The value of q-parameter (given by $2\theta_C$) shared by the two applications in case of octet hadrons gives a guess for possible physical reason for such a connection (recall also the well-known fact that generating elements of $U_q(su_n)$ admit realization in terms of q-deformed oscillators [24]). Future research possibly involving noncomutative geometry in extra dimensions should give ultimate answer.

Acknowledgements. I express my gratitude to J. Lukierski, A. Isaev for valuable discussion of the results, to D. Anchishkin for collaboration concerning the topics in sec. 9. My sincere thanks are to D. Sorokin for the invitation to participate in this nice and useful conference.

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